Dynamical phase transition of a one-dimensional kinetic Ising model with boundaries

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(Received 13 March 2001; published 21 May 2002)

The Glauber model on a one-dimensional lattice with boundaries (for the ferromagnetic and antiferromagnetic cases) is considered. The large-time behavior of the one-point function is studied. It is shown that at any temperature, the system shows a dynamical phase transition. The dynamical phase transition is controlled by the rate of spin flip at the boundaries, and is a discontinuous change of the derivative of the relaxation time towards the stationary configuration.

DOI: 10.1103/PhysRevE.65.056129 PACS number(s): 05.50.+q, 82.40.Bj, 02.50.Ga, 05.40.−a

I. INTRODUCTION

The principles of equilibrium statistical mechanics are well established. But thermal equilibrium is a special case, and little is known about the properties of systems not in equilibrium, for example, about the relaxation towards the stationary state. Some interesting problems in nonequilibrium systems are nonequilibrium phase transitions described by phenomenological rate equations, and the way the system relaxes to its steady state. As mean-field techniques generally do not give correct results for low-dimensional systems, people are motivated to study exactly solvable stochastic models in low dimensions. Moreover, solving one-dimensional systems should, in principle, be easier. Exact results for some models on a one-dimensional lattice have been obtained, for example, in Refs. [1–14]. Different methods have been used to study these models, including analytical and asymptotic methods, mean-field methods, and large-scale numerical methods.

The Glauber dynamics was originally proposed to study the relaxation of the Ising model near equilibrium states. It was also shown that there is a relation between the kinetic Ising model at zero temperature and the diffusion annihilation model in one dimension. There is an equivalence between domain walls in the Ising model and particles in the diffusion annihilation model. Kinetic generalizations of the Ising model, for example, the Glauber model or the Kawasaki model, are phenomenological models and have been studied extensively [15–20]. Combination of the Glauber and the Kawasaki dynamics has also been considered [21–23].

In Ref. [24], an asymmetric generalization of the zero-temperature Glauber model on a lattice with boundaries was introduced. It was shown there that, in the thermodynamic limit, when the lattice becomes infinite, the system shows two kinds of phase transitions. One of these is a static phase transition, the other a dynamic one. The static phase transition is controlled by the reaction rates, and is a discontinuous change of the behavior of the derivative of the stationary magnetization at the end points, with respect to the reaction rates. The dynamic phase transition is controlled by the spin flip rates of the particles at the end points, and is a discontinuous change of the relaxation time towards the stationary configuration. Other generalizations of the Glauber model consist of, for example, alternating isotopic chains and alternating bound chains (see Ref. [25], for example). People have also considered phase transitions induced by boundary conditions (see Refs. [26–28], for example).

The scheme of the paper is as follows. In Sec. II, the model is introduced, the rates are determined using the detailed balance, and the steady state configuration of the magnetization is obtained. In Sec. III, the dynamical phase transition of the system is investigated, and it is shown that it does show a dynamical phase transition at any temperature.

II. KINETIC ISING MODEL ON A ONE-DIMENSIONAL LATTICE WITH BOUNDARIES

The model being addressed is the Glauber model on a one-dimensional lattice with boundaries. In the Glauber model, the interaction is between three neighboring sites. Spin flip brings the system into equilibrium with a heat bath at temperature T. A spin is flipped with a rate $\mu=1-\tanh(2\beta J)$ whenever the spins of both of its neighboring sites are in the same direction and is flipped with a rate $\lambda=1+\tanh(2\beta J)$ whenever the spins of both of its neighboring sites are in the opposite direction. [Here $\beta=1/(k_{\mathrm{B}}T)$.] At domain boundaries, spins are flipped with unit rate. (By the rate of any change, it is of course meant the probability of that change during the infinitesimal time interval dt divided by dt.) So the interactions can be written as

$$\uparrow\uparrow\uparrow\rightarrow\uparrow\downarrow\uparrow \quad\text{and}\quad \downarrow\downarrow\downarrow\rightarrow\downarrow\uparrow\downarrow \qquad \text{with rate } \mu,$$

$$\uparrow\downarrow\uparrow\rightarrow\uparrow\uparrow\uparrow \quad\text{and}\quad \downarrow\uparrow\downarrow\rightarrow\downarrow\downarrow\downarrow \qquad \text{with rate } \lambda,$$

$$\uparrow\uparrow\downarrow\rightleftharpoons\uparrow\downarrow\downarrow \quad\text{and}\quad \downarrow\downarrow\uparrow\rightleftharpoons\downarrow\uparrow\uparrow \qquad \text{with rate } 1.$$

Consider a lattice with L sites and the Glauber dynamics as the interaction. The spin of the first site may flip with the following rates:

$$\uparrow\downarrow\rightarrow\downarrow\downarrow \qquad \text{with rate } g_1,$$

$$\uparrow\uparrow\rightarrow\downarrow\uparrow \qquad \text{with rate } g_2,$$

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$$\downarrow\uparrow\rightarrow\uparrow\uparrow \quad \text{with rate } g_3,$$

$$\downarrow\downarrow\rightarrow\uparrow\downarrow \quad \text{with rate } g_4,$$

and the spin of the last site may flip with the following rates:

$$\downarrow\uparrow\rightarrow\downarrow\downarrow \quad \text{with rate } h_1,$$

$$\uparrow\uparrow\rightarrow\uparrow\downarrow \quad \text{with rate } h_2,$$

$$\uparrow\downarrow\rightarrow\uparrow\uparrow \quad \text{with rate } h_3,$$

$$\downarrow\downarrow\rightarrow\downarrow\uparrow \quad \text{with rate } h_4.$$

It is known that the time evolution equations for the one-point functions in the bulk are expressed in terms of only the one-point functions [15]. To make this true for the boundaries as well, the following relations should hold:

$$g_1+g_4=g_2+g_3,$$

$$h_1+h_4=h_2+h_3. \tag{1}$$

One may give a physical meaning to the parameters g_i and h_i, by demanding the detailed balance to hold. Consider the energy $\mathcal{E}$ of the system to be

$$\mathcal{E}=-\left(B_1s_1+B_Ls_L+J\sum_{i=1}^{L-1}s_is_{i+1}\right), \tag{2}$$

then, the detailed balance demands

$$\mathcal{R}(s_1s_2\rightarrow s_1's_2)\exp\{\beta(B_1s_1+Js_1s_2+\cdots)\} =\mathcal{R}(s_1's_2\rightarrow s_1s_2)\exp\{\beta(B_1s_1'+Js_1's_2+\cdots)\}, \tag{3}$$

where $\mathcal{R}(s_1s_2\rightarrow s_1's_2)$ is the rate of the spin flip of the first site from s_1 to s_1'. Equation (3) shows that

$$\mathcal{R}(s_1s_2\rightarrow s_1's_2)=f(s_2)\exp\{-\beta s_1(B_1+Js_2)\}. \tag{4}$$

The exponential term in the above equation is at most linear in terms of s_1. So,

$$\mathcal{R}(s_1s_2\rightarrow s_1's_2)=\overline{f}(s_2)[1-s_1\tanh\beta(B_1+Js_2)]. \tag{5}$$

Then,

$$g_1=\overline{f}(-1)[1-\tanh\beta(B_1-J)],$$

$$g_2=\overline{f}(1)[1-\tanh\beta(B_1+J)],$$

$$g_3=\overline{f}(1)[1+\tanh\beta(B_1+J)],$$

$$g_4=\overline{f}(-1)[1+\tanh\beta(B_1-J)]. \tag{6}$$

The condition of exact solvability (1) (the closure of time evolution equation of one-point functions) leads to

$$\overline{f}(1)=\overline{f}(-1). \tag{7}$$

This means that the *inertia* of the first spin against spin flip does not depend on the second spin. A similar expression can be written for the rate of the spin flip of the last site.

For the infinite lattice, the Glauber model has a particle reaction-diffusion interpretation. If the spins of the neighboring sites are different (at a domain wall), one may consider the link between those sites as a particle. When the spins of the neighboring sites are the same (no domain wall), one may consider the link between the sites as a vacancy. Then the Glauber model turns into a reaction-diffusion model:

$$\bullet\,\bullet\rightarrow\circ\,\circ \quad \text{with rate } 1+\tanh(2\beta J),$$

$$\circ\,\circ\rightarrow\bullet\,\bullet \quad \text{with rate } 1-\tanh(2\beta J),$$

$$\bullet\,\circ\rightleftharpoons\circ\,\bullet \quad \text{with rate } 1,$$

where a particle (a vacancy) is denoted by $\bullet$ ($\circ$). For the Glauber model with boundaries, to have a consistent particle model, one has to impose

$$g_1=g_3, \quad g_2=g_4,$$

$$h_1=h_3, \quad h_2=h_4. \tag{8}$$

Then, the injection and extraction of particles at the first site are

$$\bullet\rightarrow\circ \quad \text{with rate } g_1=g_3,$$

$$\circ\rightarrow\bullet \quad \text{with rate } g_2=g_4, \tag{9}$$

and the injection and extraction of particles at the last site are

$$\bullet\rightarrow\circ \quad \text{with rate } h_1=h_3,$$

$$\circ\rightarrow\bullet \quad \text{with rate } h_2=h_4. \tag{10}$$

Now, consider the general case where only the conditions (1), that guarantee the closure of the time evolution, are satisfied. We have

$$\langle\dot{s}_k\rangle=-2\langle s_k\rangle+(\langle s_{k+1}\rangle+\langle s_{k-1}\rangle)\tanh(2\beta J), \quad 1<k<L,$$

$$\langle\dot{s}_1\rangle=-(g_2+g_3)\langle s_1\rangle+(g_1-g_2)\langle s_2\rangle+(g_3-g_1),$$

$$\langle\dot{s}_L\rangle=-(h_2+h_3)\langle s_L\rangle+(h_1-h_2)\langle s_{L-1}\rangle+(h_3-h_1). \tag{11}$$

The steady-state solution to Eq. (11) is

$$\langle s_k\rangle=D_1z_1^k+D_2z_2^{k-L-1}, \tag{12}$$

where

$$z_1=z_2^{-1}=\tanh(\beta J). \tag{13}$$

It can be shown that in the thermodynamic limit ($L\rightarrow\infty$),

$$D_1=\frac{g_1-g_3}{(g_1-g_2)z_1^2-(g_2+g_3)z_1},$$

$$D_2=\frac{h_1-h_3}{(h_1-h_2)z_1^2-(h_2+h_3)z_1}.$$

D_1 and D_2 are continuous functions of the rates. So the behavior of $\langle s_k\rangle$ near the ends of the lattice varies continuously with rates, and there is no static phase transition.

III. THE DYNAMICAL PHASE TRANSITION OF THE SYSTEM

The average magnetization per site $m(t)$ is

$$m(t)=\frac{1}{L}\sum_{k=1}^{L}\langle s_k(t)\rangle. \tag{14}$$

In the thermodynamic limit, the boundary terms are negligible, and

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t)=2[\tanh(2\beta J)-1]m(t). \tag{15}$$

Then, similar to the case of the Glauber model on an infinite lattice, the average magnetization does not show any phase transition. But, as it will be shown, the system does exhibit dynamical phase transition.

The homogeneous part of Eq. (11) can be written as

$$\langle \dot{s}_k\rangle=h_k^l\langle s_l\rangle. \tag{16}$$

The eigenvalues and eigenvectors of the operator h satisfy

$$\begin{aligned}
E\,x_k&=-2x_k+\tanh(2\beta J)(x_{k+1}+x_{k-1}),\quad k\neq 1,L,\\
E\,x_1&=-(g_2+g_3)x_1+(g_1-g_2)x_2,\\
E\,x_L&=-(h_2+h_3)x_L+(h_1-h_2)x_{L-1},
\end{aligned} \tag{17}$$

where the eigenvalue and the eigenvector have been denoted by E and x, respectively. The solution to these is

$$x_k=az_1^k+bz_2^k, \tag{18}$$

where

$$\begin{aligned}
&-(E+g_2+g_3)(az_1+bz_2)+(g_1-g_2)(az_1^2+bz_2^2)=0,\\
&-(E+h_2+h_3)(az_1^L+bz_2^L)+(h_1-h_2)(az_1^{L-1}+bz_2^{L-1})=0,
\end{aligned} \tag{19}$$

and z_j's satisfy

$$E=-2+\tanh(2\beta J)(z+z^{-1}). \tag{20}$$

So, $z_1z_2=1$. Using this and Eq. (20), one can eliminate E, and arrive at

$$\begin{aligned}
&z^{1-L}\{2-g_2-g_3+z[g_1-g_2-\tanh(2\beta J)]-z^{-1}\tanh(2\beta J)\},\\
&\quad\times\{2-h_2-h_3+z[h_1-h_2-\tanh(2\beta J)]\\
&\quad-z^{-1}\tanh(2\beta J)\},-z^{L-1}\{2-g_2-g_3+z^{-1}[g_1-g_2\\
&\quad-\tanh(2\beta J)]-z\tanh(2\beta J)\},\\
&\quad\times\{2-h_2-h_3+z^{-1}[h_1-h_2-\tanh(2\beta J)]\\
&\quad-z\tanh(2\beta J)\}=0.
\end{aligned} \tag{21}$$

Obviously, $z_j=\pm 1$ satisfies Eq. (21). But these solutions lead to

$$x_k=z^k(a+bk). \tag{22}$$

And this form for x_k generally does not satisfy the boundary conditions at $k=1,L$. Equation (21) can be written in the form

$$G(z)=F(z)-F(z^{-1})=0, \tag{23}$$

where

$$\begin{aligned}
F(z)=z^{1-L}&\{2-g_2-g_3+z[g_1-g_2-\tanh(2\beta J)]\\
&-z^{-1}\tanh(2\beta J)\}\times\{2-h_2-h_3+z[h_1-h_2\\
&-\tanh(2\beta J)]-z^{-1}\tanh(2\beta J)\}.
\end{aligned} \tag{24}$$

For a phase solution to Eq. (21), $z=e^{i\vartheta}$, we have

$$E=-2+2\tanh(2\beta J)\cos\vartheta. \tag{25}$$

In the thermodynamic limit ($L\to\infty$), in any neighborhood of $z=1$ there exists a phase solution to Eq. (21). The supermum of the eigenvalues determines the relaxation time toward the stationary average-density profile. So, if all of the solutions are phase,

$$\tau=[-2+2\tanh(2\beta J)]^{-1}. \tag{26}$$

Suppose now that there exist solutions that are not phases. Consider $|z|>1$. Then for $L\to\infty$, Eq. (21) becomes

$$\begin{aligned}
&\{2-g_2-g_3+z^{-1}[g_1-g_2-\tanh(2\beta J)]-z\tanh(2\beta J)\}\\
&\quad\times\{2-h_2-h_3+z^{-1}[h_1-h_2-\tanh(2\beta J)]\\
&\quad-z\tanh(2\beta J)\}=0.
\end{aligned} \tag{27}$$

In general, equation complex solutions. First assume that the solutions are real. Changing the rates, one may arrive at a situation where the above equation has a real solution greater than 1. The transition occurs at the point where this equation has a solution equal to 1. When the system has passed this point, the relaxation time becomes

$$\tau=[-2+2(\Lambda+\Lambda^{-1})\tanh(2\beta J)]^{-1}, \tag{28}$$

where Λ is that solution to Eq. (27), which is greater than 1. [Here we have assumed $J>0$, the ferromagnetic case. For the antiferromagnetic case $J<0$; Λ is that solution to

Eq. (27) which is less than -1.] Putting $z=1$ in Eq. (27), at least one of the following equations should hold:

$$2[1-\tanh(2\beta J)]-g_2-g_4=0,$$

$$2[1-\tanh(2\beta J)]-h_2-h_4=0. \tag{29}$$

If the temperature is zero, Eq. (29), for example, gives $g_2+g_4=0$. Remembering that these parameters are rates, one arrives at $g_2=g_4=0$. So, at zero temperature, the solution cannot pass $z=1$. But at any other temperature, $1-\tanh(2\beta J)$ is positive, and changing the parameters, g_2+g_4 can be made more than or less than $1-\tanh(2\beta J)$.

If one uses the expressions (6) and (7) for g_i's, then Eq. (29) becomes

$$-2\tanh(2\beta J)+\overline{f}[\tanh\beta(J-B_1)+\tanh\beta(J+B_1)]=0. \tag{30}$$

Putting $\overline{f}=1$, the *inertia* of the first spin against the spin flip is the same as those of the bulk spins. In this case, however, Eq. (30) has no solution. That is, there is no phase transition. In fact, Eq. (30) has no solution for $\overline{f}\leqslant 1$. For $\overline{f}>1$, however, it may have a solution.

It is seen that the parameters g_2 and g_4 (or h_2 and h_4) are control parameters of the dynamical phase transition. The parameters g_1 and g_3 (or h_1 and h_3) do not have any contribution in the dynamical phase transition. The rates g_1 and g_3 are the rates of the disappearance of the domain walls. But we note that the eigenvector corresponding to $z=1$ is a configuration where all the spins are the same ($s_k\sim z^k=1$.) It is this configuration that corresponds to the largest value of E, which determines the relaxation time, and in this configuration, there is no domain wall. The disappearance rate of this configuration determines the relaxation time towards the steady state, and g_1 and g_3 (or h_1 and h_3) are irrelevant to this rate. In the particle-vacancy picture, this means that the rate of change of vacancy to particle is important, since the configuration corresponding to the maximum value of E is the empty lattice.

This argument is true for $J>0$, the ferromagnetic case. If $J<0$, then the relaxation time is determined by the value of E at the smallest possible value of z (which is less than -1), and the transition occurs as $z=-1$ becomes a solution to Eq. (27). It is not difficult to see that in this case g_1+g_3 (or h_1+h_3) determines the phase transition. The reasoning is the same as above, except that here the configuration determining the relaxation time is that corresponding to $z=-1$, which means that the spins are alternating. So, in this configuration there are no $\uparrow\uparrow$ or $\downarrow\downarrow$ configurations and g_2 and g_4 (or h_2 and h_4) are irrelevant.

Now consider the general case where the solutions to Eq. (27) are complex. Considering the expression for E in terms of z, it is seen that there may be a larger value for $\mathrm{Re}(E)$ if

$$\mathrm{Re}(z+z^{-1})>2. \tag{31}$$

(This is for the ferromagnetic case, $J>0$.) Putting $z=X+iY$, it is seen that the boundary of the two phases is

$$Y=\pm(X-1)\sqrt{\frac{X}{2-X}}\ . \tag{32}$$

The roots of the first bracket of Eq. (27) satisfy

$$(2-g_2-g_3)X+g_1-g_2-\tanh(2\beta J)-(X^2-Y^2)\tanh(2\beta J)=0,$$

$$[2-g_2-g_3-2X\tanh(2\beta J)]Y=0. \tag{33}$$

The root $Y=0$ of the second equation corresponds to the real solutions previously considered. There exists, however, another solution. Using this solution, and Eq. (32), one arrives at

$$[4\tanh(2\beta J)-2+g_2+g_3][g_1-g_2-\tanh(2\beta J)]+(2-g_2-g_3)\tanh(2\beta J)=0. \tag{34}$$

Specifically, at zero temperature we have

$$(g_1-g_2)(g_2+g_3)-2(g_2+g_4)=0, \tag{35}$$

where Eq. (1) has been used. It is clear that this boundary can be passed (as an example, one can consider the special case $g_1=g_3$ and $g_2=g_4$), so that the system does have a dynamical phase transition even at zero temperature.

For the antiferromagnetic case ($J<0$), equations corresponding to (34) and (35) are

$$[g_1-g_2-\tanh(2\beta J)][2-g_2-g_3+4\tanh(2\beta J)]-(2-g_2-g_3)\tanh(2\beta J)=0, \tag{36}$$

and

$$(g_4-g_3)(g_2+g_3)-2(g_3+g_1)=0, \tag{37}$$

respectively, and it is easily seen that there is a phase transition even at zero temperature.

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